Interactions of time and technology as critical determinants of optimal climate change policy

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Abstract

We derive the optimal climate policy for a social planner who aims to minimize the sum of climate damages and adjustment costs. Our stylized model focuses on the dynamic structure of abatement costs, which reflects interactions of time and technology. Specifically, we highlight the importance of inertia, induced innovation, and path dependency. Our analytic solution shows that the degree of dependence on such characteristics has important implications for the optimal policy, the optimal abatement effort ?today,? and the costs associated with delaying such efforts.
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1 Introduction

As concern over climate change has grown, governments' goals, as reflected in the COP21 agreement in Paris in 2015, have become increasingly ambitious. National targets to reach "net zero" emissions – required if global temperatures are to be stabilized – now cover all major economies and a large share of global emissions (IPCC, 2022). Such ambitious mitigation goals – often involving major and potentially rapid sectoral transformations – raise important questions about how net zero emissions targets can be achieved (Stern, 2022).

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It is therefore pertinent to develop models that consider the *timing* of abatement. In this paper, we develop a stylized and analytically tractable model that formalizes important dynamic aspects of abatement costs, focusing on inertia, induced innovation, and path dependency. We posit that abatement costs consist of two components: a rigid and a transitional one. The former refers to traditional "static" abatement costs. The latter refers to temporary, transitional costs, which we break down into two subcomponents: one associated with inertia and one with induced innovation. The transitional cost associated with inertia is defined as the cost of changing pathways, which we call the characteristic time of the system. The transitional cost associated with induced innovation consists of investments that yield long-term streams of emissions reductions beyond the directly amortized costs; for example, by inducing low-carbon innovation or long-lived infrastructure. As a consequence of induced innovation, abatement in one period reduces the rigid cost component in subsequent periods. Inertia, together with induced innovation, creates path dependence. The degree to which a system depends on transitional relative to rigid costs can thus be seen as representing the degree of path dependency of the system.

We are interested in how the relative proportions of rigid and transitional costs affect the (optimal) behavior of the system. We model optimal long-run emissions and consider different regimes to achieve those emissions: one with purely rigid costs, one with moderate transitional costs, and one with predominantly transitional costs.

Our analytic solution shows that a system with purely rigid abatement costs is akin to that in the standard DICE model, which implies a sudden jump in abatement at time zero (and hence in annual emissions). Intuitively, with purely rigid costs, it makes sense to abate up to the cost value that is justified by climate damages, mediated by a carbon price. In regimes with positive transitional costs, annual emissions remain continuous at time zero, avoiding a jump in annual emissions. Because of the transitional costs, however, abatement efforts at time zero are generally higher than in the regime with purely rigid costs. In the moderate transitional cost scenario, optimal long-run emissions are reached by gradually cutting emissions to this level. The intuition here is that with transitional costs, it makes sense to transform the economy at a steady rate. With a fixed abatement target (e.g., net zero), this implies starting as soon as possible to minimize the rate of change. It follows, as we demonstrate analytically, that the abatement effort at time zero typically exceeds that in the regime with purely rigid costs because a substantial part of the abatement effort involves transforming the emitting systems, the results of which are not immediately visible in the marginal emissions path. In the third regime, with predominantly transitional costs, the pace at which emissions can be lowered is effectively constrained. Since "steering" the system on the "right path" is expensive, it can be optimal to overshoot the long-term goal (i.e., to "oversteer") before later correcting or "steering back", implying that optimal emissions oscillate (with diminishing amplitude) toward the long-term optimum.

The analytic nature of the solution allows us to derive further insights for the optimal abatement effort at time zero ("today") as well as the costs incurred by delaying action to combat climate change. For a scenario with purely rigid costs, we show analytically that the optimal effort today is exceedingly sensitive to the discount rate. We interpret this to mean that with no transitional costs in the system, but in the presence of climate damages, the optimal emissions path takes a point of departure that is substantially different from current-day emissions. At low discount rates, the optimal solution suggests an immediate, large jump in the current level of annual emissions, which is further amplified to counteract the impact of any rising trend of emissions. Introducing transitional costs introduces two opposing forces (inertia and induced innovation) which together create path dependence. Given induced innovation, it makes sense to start abatement as early as possible, increasing today's effort. Inertia, however, has the opposite effect. In a system with low (high) inertia, the cost of overcoming inertia is lower than in a high (low) inertia system. Thus, the gains of induced innovation, can (cannot) easily be reaped in a system with low (high) inertia. For the cost of delay, which we define to be the change in net present value when the optimal response is delayed by a short amount of time $\mathrm{d}t$, we show that our optimal solution is highly sensitive to deviations from the optimal policy. Indeed, the cost of delay far outweighs the optimal initial effort.

Related literature. Our paper ties in with the large body of literature on integrated assessment models (IAMs) of climate change (e.g., Nordhaus, 1991, 1993; Golosov et al., 2014; van der Ploeg and Rezai, 2019; Weyant, 2017; Harmsen et al., 2021; Stern and Stiglitz, 2022). While complex process-based IAMs have been used to provide detailed disaggregated estimates of abatement costs with growing attention for capital stock and innovation (IPCC, 2007, 2014, 2022), our goal is to use a stylized model to obtain analytical insights into how such dynamics affect optimal abatement. We build on the intuition in Grubb et al. (1995) to develop an analytically tractable model to evaluate the optimal balance between changes in temperature driven by cumulative emissions (Ricke and Caldeira, 2014; Mattauch et al., 2020) and three dynamic features of emissions mitigation.

The first of these features is induced innovation. This includes endogeneity both in innovation between high and low carbon technologies (Acemoglu et al., 2012, 2014; Aghion and Jaravel, 2015; Grubb et al., 2021) and in economic systems more widely (Gillingham et al., 2008; Dietz and Stern, 2015). The second feature concerns inertia. Many macroeconomic models focus on the long run and thus assume a relatively high elasticity of substitution between "green" and "dirty" technologies (e.g., Acemoglu et al., 2016; Hassler et al., 2020). This becomes problematic in view of the typically long timescales of emitting capital stock (Pottier et al., 2014) and growth rates for clean technology (Wilson and Grubler, 2015). The implications of inertia are underlined by capital stock modeling (Vogt-Schilb et al., 2018), and assume great importance in the face of higher damage costs (Howard and Sterner, 2017) and in meeting the goals of the Paris Agreement (IPCC, 2022). Finally, these two features combine to create the third feature: path dependency in emitting systems (Aghion et al., 2016, 2019).

Our model is simpler than any of these complex models. However, it is also more flexible: by focusing on the overall balance between rigid and transitional costs, it encompasses the implications of inertia, induced innovation, and path dependency together, and yields policy insights for the transition towards a net zero world.

The paper is organized as follows. Section 2 presents the main model. Section 3 states and discusses the analytical solution and provides a simulation of said solution for calibrated parameter values. Section 4 analytically calculates the optimal abatement efforts at time zero ("today"), while section 5 calculates the cost of delaying action. Section 6 concludes.

2 The model

2.1 High-level optimization problem

We define cumulative emissions at time t, relative to pre-industrial times, as $E(t)$ measured in gigatonnes of carbon (GtC). We take $t = 0$ to mean today. The historical path of $E(t)$, i.e., for $t \leq 0$, is fixed and cannot be changed. This means that $E(t) = E_{\text{ref}}(t)$ for $t \leq 0$, where $E_{\text{ref}}(t)$ is a reference trajectory that matches historical cumulative emissions for $t \leq 0$. The cumulative emissions to date are fixed at $E_0 := E_{\text{ref}}(0)$.

Going forward, i.e., for time $t > 0$, $E_{\text{ref}}(t)$ represents a "business as usual" scenario absent any substantial abatement effort. This trajectory is suboptimal in the context of climate change, such that $E(t)$ will optimally diverge from $E_{\text{ref}}(t)$ for $t > 0$. For notational simplicity, annual emissions are denoted by $e(t) := E'(t)$, measured in GtC per year (GtC/yr). The reference trajectory of annual emissions is written as $e_{\text{ref}}(t) = E'_{\text{ref}}(t)$. The constant $e_{\text{ref}}(0) = e_0$ represents current-day emissions.

The stylized high-level problem we are interested in solving is

$$\min_{\{E(t)\}_{t=0}^{t=T}} \int_0^T \exp(-r\,t)\, F[E(t), e(t), e'(t)]\, \mathrm{d}t, \tag{1}$$

$$\text{s.t.} \qquad E(0) = E_0, \tag{2}$$

$$\text{and} \qquad e(0) = e_0, \quad \text{(this restriction is optional)}. \tag{3}$$

Here $\{E(t)\}_{t=0}^{t=T}$ denotes the path of cumulative emissions $E(t)$ from $t = 0$ to $t = T$, where $T > 0$ measured in years (yr) is the time horizon, which may be infinite, $r > 0$ is the discount rate, and $F[\cdot,\cdot,\cdot]$ is a function depending on $E(t)$ and its first two derivatives, denoted $e(t)$ and $e'(t)$. The cumulative emissions path $E(t)$ for $0 \leq t \leq T$ together with the boundary conditions (2) and optionally (3) implies the annual emissions path $e(t)$ for $0 \leq t \leq T$, as well as its rate of change, $e'(t)$. This means that $E(t)$ can—without loss of generality—be used as the control variable.

The function $F[\cdot,\cdot,\cdot]$, measured in USD per year, is the sum of a climate-damage function $D[\cdot]$ and an abatement-cost function $C[\cdot,\cdot]$:

$$F[E(t), e(t), e'(t)] \;:=\; D[E(t)] \,+\, C[e(t), e'(t)]. \tag{4}$$

The damage function reflects the form in the majority of stylized IAMs, which relate climate damages to global temperature change, using the finding that this is roughly proportional to cumulative CO_2 emissions (neglecting shortlived gases). Consequently, at any given point in time, climate damages $D[\cdot]$ depend on cumulative emissions up to that point, i.e., $E(t)$ (see section 2.2 for details).

Abatement costs $C[\cdot,\cdot]$, on the other hand, depend on both annual emissions $e(t)$ and their rate of change, $e'(t)$. Classic IAM models take $C = C[e(t)]$, i.e., without dependence on $e'(t)$, such that the abatement cost at time t depends exclusively on the annual emissions at time t. As indicated, we call these rigid costs. This represents the classic structural form of an abatement cost curve. To these we add transitional costs by allowing $C[\cdot,\cdot]$ to depend additionally on $e'(t)$. As outlined, this comprises the elements of inertia and induced innovation (see section 2.3 for details).

The minimization problem (1) is subject to constraint (2), implying that cumulative emissions $E(t)$ must be continuous at $t = 0$: we cannot instantly extract carbon from the atmosphere. Many models, including DICE, optimize annual emissions in a given period (including at time zero): consequently, there can be discontinuities in annual emissions when climate damages are introduced, and steep reductions if a low-carbon technology suddenly becomes competitive. With transitional dynamics, however, such jumps in global emissions are implausible (and very costly). Constraint (3) implies that $E(t)$ smoothly matches the reference trajectory at $t = 0$, by making annual emissions $e(t)$, too, continuous at $t = 0$. To maintain comparability with standard models without inertia, this constraint is optional.

2.2 Climate-damage function

A central estimate is that global temperatures increase by 1 degree Celsius with each additional 600 GtC in cumulative emissions (IPCC, 2021, Table SPM.2). In line with much of the stylized literature, including the common default assumption in DICE, we assume that global damages increase quadratically with temperature. The climate-damage function, $D[\cdot]$, is thus simply:

$$D[E(t)] \; = \frac{d}{8} E(t)^2, \tag{5}$$

where $d > 0$ is a damage parameter with dimensions USD/(yr$\times$GtC^2), such that damages have a dimension of USD/yr. The numerical factor 1/8 is arbitrary and chosen for later convenience. Damage function (5) ignores any time lag between emissions reductions and its equilibrium impact on temperature. Contrary to common assumptions, this time lag is relatively small: Ricke and Caldeira (2014) estimate the median time lag (until maximum warming occurs) to be just over 10 years (see also Mattauch et al., 2020).

2.3 Abatement-cost function

We specify the abatement cost function, $C[\cdot, \cdot]$, in terms of abatement $a(t)$, and its rate of change $a'(t)$ as follows:

$$C[e(t), e'(t)] := \; c\left[q\, a(t)^2 \; + \; 2\, p\, \tau^2\, a'(t)^2\right], \tag{6}$$

$$\text{where } a(t) := \; e_{\text{ref}}(t) - e(t). \tag{7}$$

Here $c > 0$ is an overall cost-scaling constant, measured in USD $\times$ yr/GtC^2, and the numerical factor of 2 in the second term of equation (6) is arbitrary but included for later convenience. Abatement at time t, $a(t)$ is measured in GtC/yr relative to baseline, while $a'(t)$, in GtC/yr^2, represents its rate of change. The resulting cost function expresses annual expenditure on emissions abatement in USD/yr.

The first term in equation (6) captures the traditional stylized formulation of abatement costs as a nonlinear function of the degree of abatement relative to a baseline projection, scaled by $q \in [0, 1]$, a dimensionless number. In common with several other models, we assume that the rigid abatement cost at time t increases quadratically with the abatement effort at time t, giving rise to a term that scales with $a(t)^2$.[1]

[1] Nordhaus (2013) has $a(t)^{2.8}$. Since Grubb et al. (2018) show that learning-by-doing tends to reduce the scale and the convexity of the marginal cost curve, we use $a(t)^2$.

The second term captures the transitional cost, which is proportional to the square of the rate of change of abatement and measures how rapidly the system is forced to deviate from the reference trajectory.[2] Here we introduce two parameters, τ and p. $\tau > 0$, measured in years, reflects the intrinsic inertia of the system in terms of a characteristic transition time: the higher τ, the longer it takes to achieve a given level of abatement for a given cost (or the more difficult it is to overcome this inertia). $p \in [0, 1]$ reflects the "pliability" of the system, with $q = 1 - p$ being its complement. We use p to explore the implications of some portion of costs being transitional that otherwise, in a DICE-like framework, would have been attributed to rigid costs.

Inertia and induced innovation together create path dependence. Given our interest in how the relative scale of rigid and transitional costs affect the behavior of the system, the ratio p/q can be taken to represent the degree of path dependency of the system: if p is very high (i.e., close to 1), then after transitional abatement in one period, the system will tend to stick to its new trajectory.

2.4 Business-as-usual scenario

To complete the model setup, the reference path of cumulative emissions is specified by assuming linear growth in annual emissions as follows:

$$e_{\text{ref}}(t) := e_0 + e_1\, t, \quad t \geq 0, \tag{8}$$

$$E_{\text{ref}}(t) = E_{\text{ref}}(0) + \int_0^t e_{\text{ref}}(s)\, \mathrm{d}s = E_0 + e_0\, t + \frac{e_1}{2}\, t^2, \quad t \geq 0, \tag{9}$$

where $e_0 \geq 0$ is the annual emissions at $t = 0$, while $e_1 \geq 0$, measured in GtC/yr^2, represents a (linear) growth rate of annual emissions assuming "business as usual." We take as our reference scenario a view in which global emissions rise at moderate pace of $e_1 = 120$ MtC/yr^2 (0.12 GtC/yr^2), approximating the average trend since the financial crisis. Problem (1) has now been specified in its entirety. Table 1 contains an overview of all symbols used as well as their dimensions.

3 Analytic solution

3.1 Statement of the solution

Optimization problem (1) permits an analytic solution as described here.

Theorem 1 *Consider optimization problem* (1) *with infinite time horizon ($T = \infty$) and subject to equations* (4) *through* (9)*. The general form of the solution is*

$$E(t) \;=\; E_\star \,+\, e_\star \cdot t \,+\, \sum_{j=1}^{2} \, Z_j\, \exp\left(z_j\, t/2\right), \quad t \geq 0, \tag{10}$$

[2]Though there is less evidence on the functional form of transitional costs, they are clearly convex (see Grubb et al., 2018). Note that the quadratic form is the same as the one assumed for the cost of accelerating renewables expansion in REMIND (Bauer et al., 2016).

Symbol	Meaning	Dimension	Calibration
p, q	pliability and its complement $q = 1 - p$	none	
t, T	time, time horizon	yr	
τ	characteristic transition time	yr	15
r	discount rate	1/yr	0.025
$C[\cdot,\cdot], D[\cdot]$	abatement cost and climate damage	USD/yr	
c	abatement cost parameter	USD $\times$ yr/GtC^2	0.026
d	damage parameter	USD/(yr $\times$ GtC^2)	0.00002
$E(t), E_{\text{ref}}(t)$	cumulative emissions at time t	GtC	
E_0	cumulative emissions at $t = 0$	GtC	665
$e(t), e_{\text{ref}}(t)$	annual emissions at time t	GtC/yr	
$a(t)$	abatement at time t, $a(t) := e_{\text{ref}}(t) - e(t)$	GtC/yr	
e_0	annual emissions at $t = 0$	GtC/yr	10.4
$e'(t), e'_{\text{ref}}(t)$	rate of change of annual emissions at time t	GtC/yr^2	
$a'(t)$	rate of change of abatement at time t	GtC/yr^2	
e_1	reference growth of annual emissions	GtC/yr^2	0.12

Table 1: Overview of symbols, dimensions, and calibration

where constants $E_\star$, with dimension GtC, and $e_\star$, with dimension GtC/yr, are

$$E_\star = \frac{8\,c\,q\,(e_0\,r - e_1)}{d} + 64\,e_1\,r^2\left(\frac{c\,p\,\tau^2}{4\,d} - \left(\frac{c\,q}{d}\right)^2\right), \qquad e_\star = 8\frac{c\,q\,e_1\,r}{d}. \tag{11}$$

The form of the exponential constants z_j, with dimension $1/yr$, and Z_j, with dimension GtC, for $j = 1, 2$ depend on the pliability p of the system relative to a critical threshold $p^\star$, defined as

$$p^\star := 1 - \frac{\sqrt{1+4x} - 1}{2x} \in (0, 1),$$

where $x := c/(d\tau^2) \in (0, \infty)$ is a dimensionless characteristic of the system. The solution then comprises three distinct regimes:

1. **No pliability.** *Assume $p = 0$ and impose constraint* (2)*. Then*

$$z_1 = r - \sqrt{r^2 + \frac{d}{2\,c\,q}}, \quad Z_1 = E_0 - E_\star, \qquad z_2 = 0, \qquad Z_2 = 0. \tag{12}$$

2. **Medium pliability.** *Assume $0 < p \leq p^*$ and impose constraints* (2) *and* (3)*. Then $z_1 = z_+$ and $z_2 = z_-$, where*

$$z_\pm = r - \sqrt{v \pm \sqrt{u}} \ < \ 0, \tag{13}$$

where $u := \frac{q^2}{p^2\tau^4} - \frac{d}{cp\tau^2}$ and $v := r^2 + \frac{q}{p\,\tau^2}$. Both z_1 and z_2 are real valued and strictly negative. The exponential constants Z_j for $j = 1, 2$ are given by

$$Z_1 = \frac{2(e_0 - e_\star) + z_-(E_\star - E_0)}{z_+ - z_-} \qquad Z_2 = \frac{2(e_0 - e_\star) + z_+(E_\star - E_0)}{z_- - z_+}. \tag{14}$$

3. **High pliability.** *Assume $p > p^\star$ and impose constraints* (2) *and* (3)*. Then $z_1 = z_+$ and*

$z_2 = z_-$, where

$$z_{\pm} \;=\; r \,-\, w \,\pm\, \mathrm{i}\,\frac{\sqrt{|u|}}{2w}, \tag{15}$$

where $\mathrm{i} = \sqrt{-1}$, $w := \frac{\sqrt{v+\sqrt{v^2+|u|}}}{\sqrt{2}}$, *while u, v remain as under point 2. Both z_1 and z_1 are complex valued, with real parts that are strictly negative. Constants Z_1, Z_2 remain as in equation* (14)*, but with $z_{\pm}$ as in equation* (15)*.*

Proof: A standard application of the calculus of variations (e.g., Goldstein et al., 2013) gives a fourth-order differential equation for the solution $E(t)$. Conjecture (10) yields expressions for $E_\star$ and $e_\star$, as well as a fourth-order polynomial equation for the constants z_j for $j = 1, 2$. Two roots can be discarded because of the (implicit) boundary condition at $T = \infty$. The two remaining roots can be found analytically, giving z_1 and z_2. Depending on the regime, these are either real (no-pliability and medium-pliability regimes) or complex (high-pliability regime). The constants Z_j for $j = 1, 2$ follow from the boundary conditions at $t = 0$ and are expressable in terms of z_1 and z_2. In all cases, the cumulative emissions path $E(t)$ remains real. Details of the proof can be found in the online Appendix A.

3.2 Discussion of the solution

Equation (10) in Theorem 1 gives the optimal solution $E(t)$ for $t \geq 0$. As a sanity check, it can be verified that the limits (i) $d \to 0$ or (ii) $c \to \infty$ imply $E(t) \to E_{\mathrm{ref}}(t)$. That is, when (i) damages are zero or (ii) the abatement cost approaches infinity, the optimal path is equal to the reference path. For non-zero damages ($d > 0$) and finite abatement cost ($c < \infty$), the path of $E(t)$ lies below that of $E_{\mathrm{ref}}(t)$ for $t \geq 0$. In particular, equation (10) gives $E(t)$ as the sum of a constant $E_\star$, a linear function of time with slope $e_\star$, and a sum of two exponential functions. The (real parts of the) exponential parameters z_j for $j = 1, 2$ are negative, such that, as $t \to \infty$, these terms vanish.

Hence, Theorem 1 implies that optimal cumulative emissions are, asymptotically, linear in time with annual increase $e_\star \geq 0$. Equivalently, optimal long-run annual emissions $e(t) = E'(t)$ are constant at the level $e_\star$ given in equation (11). The positive emissions are determined by the balance of damages and abatement costs, both of which increase quadratically if reference emissions are rising. As might have been anticipated, the optimal level $e_\star$ is an increasing function of the abatement-cost parameter c, the business-as-usual emissions parameter e_1, and the discount rate r, while it is a decreasing function of the damage parameter d. The dimensions of $e_\star$ are GtC/yr. For a fully pliable system ($p = 1, q = 0$), or stable reference emissions ($e_1 = 0$), we have $e_\star = 0$, i.e., it is optimal in the long run to decarbonize entirely.

The three regimes differ in how this optimal asymptotic emissions level $e_\star$ is reached. The no-pliability solution, where $p = 0$, is akin to the standard DICE solution, and implies a sudden jump in abatement.[3] Whenever $p > 0$, i.e., for a system with any positive degree of inertia, such an immediate response is impossibly costly. Hence, in regimes 2 and 3, the path of $e(t)$ remains continuous at $t = 0$, avoiding a discontinuity in annual emissions. In the medium-pliability regime, the optimal long-run emissions level is reached by more steadily

[3] In practice, plots from DICE do not show this because emissions at time $t = 0$ are set equal to the actual emissions and the discontinuity occurs in the first unconstrained five-year period, shown as $t + 5$.

cutting emissions to this level. The abatement effort (cost) at time zero typically exceeds that in regime 1, because part of this effort is related to the transformation of the emitting systems, the results of which are not immediately visible in the marginal emissions path.

In the high-pliability regime, the exponential parameters z_j for $j = 1, 2$ are imaginary. Naturally, the cumulative emissions profile $E(t)$ remains real valued. The high-pliability response can be equivalently written in terms of trigonometric functions (sines and cosines) as follows:

$$
\begin{aligned}
E(t) &= E_\star + e_\star \cdot t \\
&\quad + \exp\left(\frac{\widehat{z}\, t}{2}\right) \left[\frac{2(e_0 - e_\star) + \widehat{z}(E_\star - E_0)}{\widetilde{z}} \sin\left(\frac{\widetilde{z}\, t}{2}\right) + (E_0 - E_\star) \cos\left(\frac{\widetilde{z}\, t}{2}\right) \right],
\end{aligned} \tag{16}
$$

where $e_\star, E_\star$ remain as in Theorem 1, while $\widehat{z}, \widetilde{z}$ are real numbers defined as $\widehat{z} := \mathrm{Re}(z_+)$ and $\widetilde{z} := \mathrm{Im}(z_+)$. The intuition for the third regime is that, when damages are high but "steering" is expensive, it might be beneficial to cut emissions at a pace which leads to some degree of "overshoot" before later correcting (steering back), which explains the appearance of trigonometric functions in the solution: emissions oscillate toward the long-term optimum.

In all three regimes, the optimal marginal emissions path $e(t)$ is implied by the optimal cumulative emissions path $E(t)$ via a straightforward differentiation with respect to time. Further, in all cases an analytic solution remains possible even for a finite optimization horizon T. The resulting, somewhat more involved, expressions are available from the authors upon request.

3.3 Numerical results

3.3.1 Calibration

This subsection describes how we calibrate the parameters of our model. An overview of the calibrated parameter values is given in Table 1.

Emissions. We define $t = 0$ to be 2019 and take values for E_0, e_0, and e_1 from the most recent IPCC report (IPCC, 2021). Specifically we set $E_0 = 665\text{GtC}$, $e_0 = 10.4\text{GtC/yr}$, and $e_1 = 0.120\text{GtC/yr}^2$.

Discount rate. We assume the real discount rate to be 2.5% per year, based on the expert elicitation survey by Drupp et al. (2018). This is a compromise between "prescriptive" and "descriptive" rates, leaning more towards the latter in that after a few decades it leads to significant discounting of costs and damages.

Climate damages. Our climate-damage estimates draw upon Nordhaus (2013) and Howard and Sterner (2017), both of which present damage estimates as proportional to the square of global temperature change, as in our model. Howard and Sterner's (2017) "preferred damage specification" is almost four times the Nordhaus (2013) value. We take a central benchmark value midway between these, resulting in $0.00002\,\text{USD}/(\text{yr} \times \text{GtC}^2)$.

Abatement costs. The vast majority of literature specifies abatement costs in terms of marginal abatement costs, some derived for specific projected years. Based on the extensive review by Harmsen et al. (2021, Figure 1) of a dozen different complex IAMs, we take an average benchmark abatement cost parameter $c = 0.026\,\text{USD} \times \text{yr}/\ \text{GtC}^2$, equivalent to a marginal abatement cost of $370\,\text{USD/tC} = 100\,\text{USD/tCO}_2$ for 50% emissions reduction from reference (7GtC/yr), in the middle of their reported range.

Characteristic time. There is little empirical literature on transition timescales. The review by Harmsen et al. (2021) introduces this metric for the first time, documenting a median value of 13.5 years across all the models they review. We thus take our benchmark value as $\tau = 15$ years.

3.3.2 Results

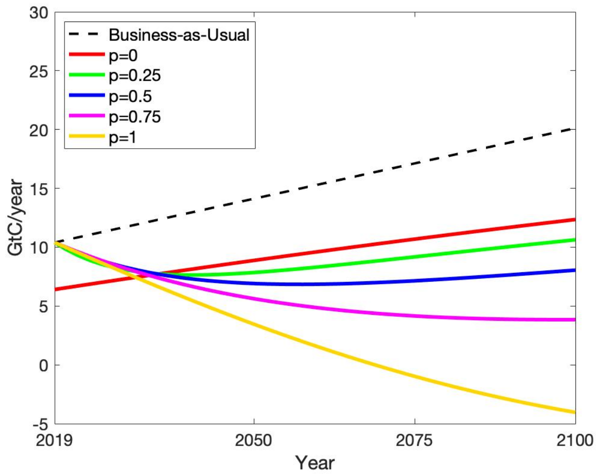
We present our main results for five different scenarios. Figure 1 displays annual emissions (in GtC per year), global mean temperature increases with respect to pre-industrial times (in degrees Celsius), annual damages from climate change (in trillion USD per year), and annual abatement costs (in trillion USD per year) for a system with no pliability (i.e., $p = 0$), a system with full pliability (i.e., $p = 1$), and three scenarios that relate to regions in between (i.e., $p \in \{0.25, 0.5, 0.75\}$).

No pliability, $p = 0$. The system with purely rigid costs resembles that used in DICE and other classical IAMs (hereafter, "classical"), as discussed in section 2. There is a prompt reduction in annual emissions (by about one third), but after this initial drop, emissions continue to rise steadily throughout the century. This is because abatement in this scenario cannot keep up with the rising emissions from the business-as-usual scenario. As can be seen, the policy recommendation implied by our model is similar to the climate-policy ramp observed in DICE. Annual abatement investment increases from below 500 billion USD per year to above 1.5 trillion USD per year in 2100. As cumulative emissions continue to rise, global mean temperatures rise above 2.5 degrees Celsius by 2100 and continue rising beyond. Damages also increase over time, reaching more than 4 trillion USD per year by the end of the century.

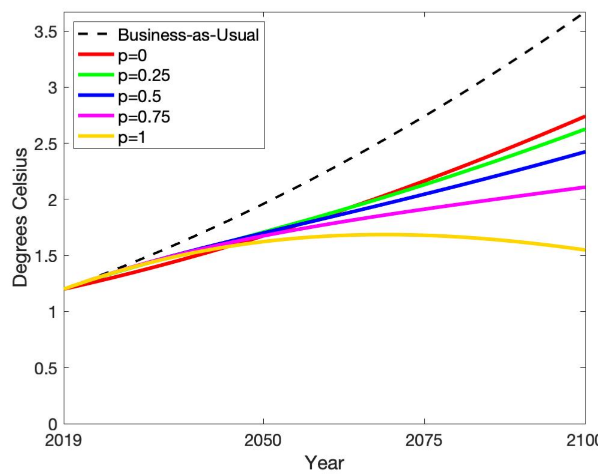
As soon as $p > 0$, an immediate emissions reduction is no longer possible. As indicated above, emissions in these positive-pliability scenarios initially decline (approximately) linearly. Once they cross below the classical $p = 0$ case, which happens after roughly $\tau = 15$ years, the behavior varies widely across the different cases.

Full pliability, $p = 1$. At the opposite extreme, the scenario with a fully pliable system is one with solely transitional costs and no rigid costs. Emissions decline steadily and reach net zero emissions around 2065. Afterwards, net annual emissions become negative, implying that cumulative emissions will decrease (as indicated for the high-pliability regime). With temperature increases proportional to cumulative emissions, the global mean temperature increases to about 1.6 degrees Celsius from pre-industrial levels at the time of net zero, and decreases slightly thereafter.

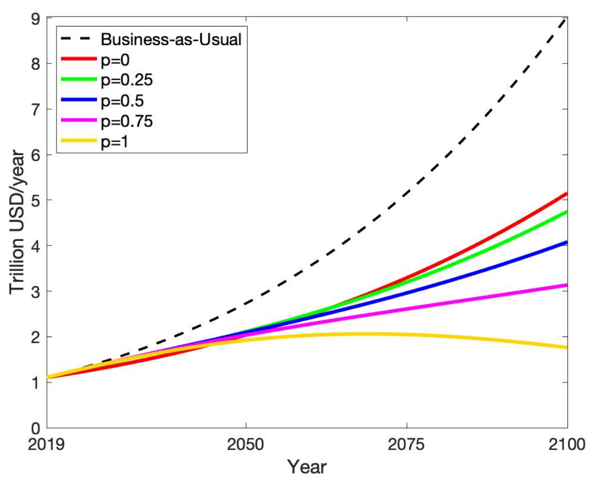
With full pliability, the optimal policy involves substantially higher initial expenditure than in the other scenarios. Initial annual abatement investment in the fully pliable system is, at over 1 trillion USD per year, almost three times greater than in the non-pliable system. In sharp contrast, optimal effort decreases rather than increases over time, reaching less than 500



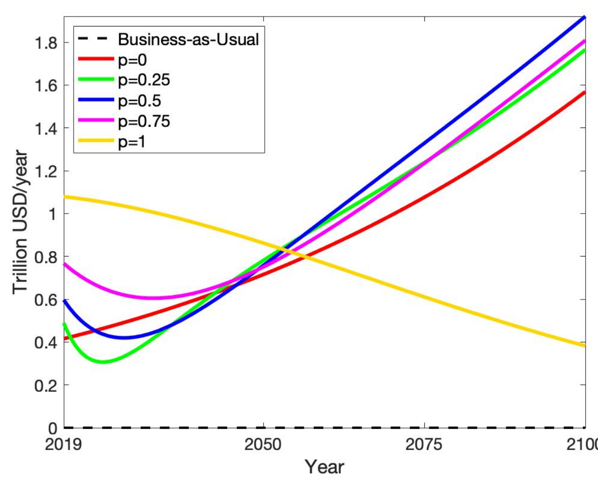
(a) Annual emissions



(b) Global mean temperature increase



(c) Annual damages



(d) Annual abatement costs

Figure 1: Optimal policy and implications for $p \in \{0, 0.25, 0.5, 0.75, 1\}$

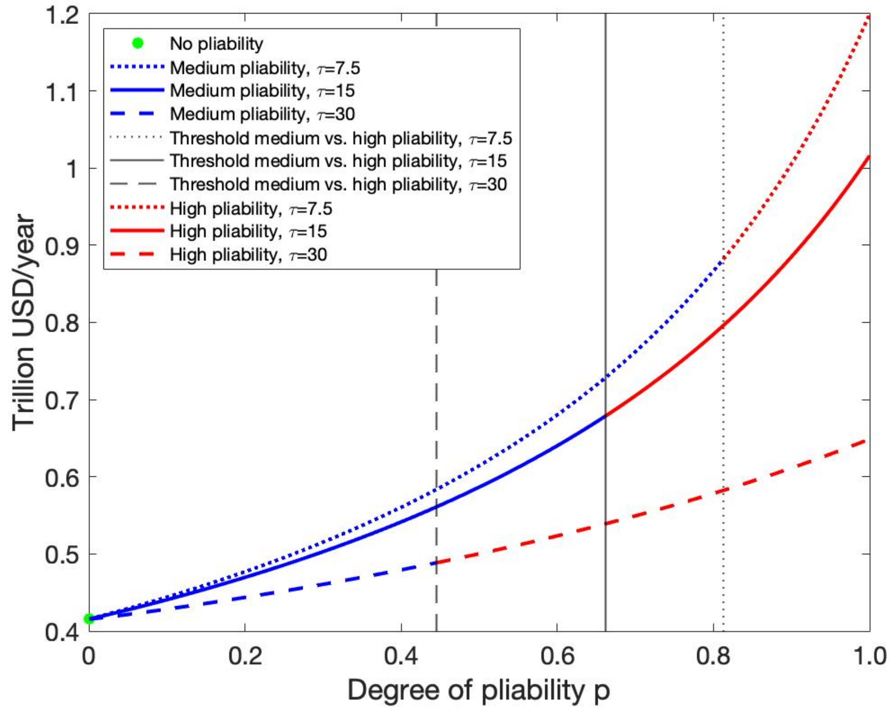


Figure 2: Optimal abatement effort at time zero from equation (18)

billion USD per year by 2100. Damages in this scenario remain much lower than in the other two cases: between one and two trillion USD per year between 2018 and 2100.

Intermediate pliability, $p = 0.25$, $p = 0.5$, $p = 0.75$. In all intermediate cases, emissions after the crossing point stay below those in the classical ($p = 0$) case, but do not reach zero; as explained above, they asymptote towards a constant level. Given the absence of a "backstop" technology, global temperatures, damages, and abatement costs all keep rising, though the $p = 0.75$ case only reaches 2 degrees Celsius towards the end of the century, precipitated by an initial doubling of the effort, which remains above the classical cost for the first half of the century, reaping the rewards in the second half with lower damages.

While damages are lower for higher p in panel (c), abatement costs in panel (d) for $p > 0$ (but $p \neq 1$) are (mostly) above the classical case with $p = 0$. The intuition is that the larger initial abatement effort in the intermediate-pliability cases leads to reduced damages later on. Given that damages are approximately three times higher than abatement costs for any given p, the reduction in damages is larger than the increase in abatement (compared to the business-as-usual scenario). Hence, for a decision maker, any $p > 0$ is to be preferred over $p = 0$, as positive pliability leads to a lower value of the (optimised) objective function (1).

4 Abatement effort at time zero

Having obtained the optimal path of cumulative emissions $E(t)$ in three regimes, we can directly compute the optimal level of abatement effort.

4.1 No inertia, no pliability regime

For $p = 0$, substituting the exact solution (10) into the cost function $C[\cdot, \cdot]$ in equation (6) and evaluating the result at $t = 0$ yields the optimal current abatement effort measured in USD/yr as follows:

$$C[e(0), e'(0)]\Big|_{p=0} = c\left[e_0 - e_\star - \frac{z_1}{2}(E_0 - E_\star)\right]^2\Big|_{p=0,q=1}, \tag{17}$$
$$= \left[\frac{e_1^2}{r^6} + \frac{2\,e_0\,e_1}{r^5} + \frac{e_0^2 + 2\,e_1\,E_0}{r^4} + \frac{2\,e_0\,E_0}{r^3} + \frac{E_0^2}{r^2}\right] \cdot \frac{d^2}{64\,c} + O(d^3),$$

where $E_\star$ and z_1 are given in equation (11) and (12), respectively. The second line is a straightforward first-order Taylor expansion in the square of the damage parameter. From this expression, it is clear that the optimal level of effort today is extremely sensitive to the discount rate r, which appears to the power of six in the denominator whenever $e_1 \neq 0$. The ratio d^2/c confirms that effort tends to increase nonlinearly with d, while higher abatement cost c suppresses effort because it reduces the benefit/cost ratio (and with discounting it is cheaper to defer the effort required). In terms of initial emission conditions, e_0, e_1, and E_0 all also increase the optimal effort.

Numerous studies with DICE have underlined sensitivity to discount rate, but to our knowledge none have identified it analytically to such a remarkable degree. Note that the first two terms in the expansion are driven by e_1, while the inverse quartic and cubic dependencies involve e_0. We interpret this as follows: without inertia, the solution suggests that in the presence of climate damages, optimal emissions today are much lower than actual emissions. At low discount rates, the solution suggests an immediate, large reduction in the starting level, which is amplified further to counteract the rising trend of future emissions. In published results from DICE and similar numeric models, this immediate reduction in annual emissions is somewhat obscured by the five-year time steps typically used, but the underlying logic is one of a sudden, potentially dramatic jump so as to “start from somewhere else.” In isolation of any consideration of dynamic constraints, it is unclear how useful this is as a policy-relevant insight, since the global energy system clearly cannot make overnight jumps in its emission levels and trajectories, as acknowledged in DICE itself (Nordhaus, 2019).

4.2 Inertia and positive-pliability regimes

For $p \neq 0$, substituting the exact solution (10) into the cost function $C[\cdot, \cdot]$ in equation (6) and evaluating the result at $t = 0$ yields the optimal current abatement effort as follows:

$$C[e(0), e'(0)] = 2\,c\,p\,\tau^2\left[e_1 - (e_0 - e_\star) \cdot \frac{z_1 + z_2}{2} - (E_\star - E_0) \cdot \frac{z_1 \times z_2}{4}\right]^2, \tag{18}$$

For a small degree of pliability (low p), the dependencies can be clarified as:

$$C[e(0), e'(0)] = C[e(0), e'(0)]\Big|_{p=0} + V \times c\,\tau\,\sqrt{2p} + O(p), \tag{19}$$

where V is a constant,[4] $e_\star$ and $E_\star$ are as in equation (11), while z_1 and z_2 are as in equation (13) (medium-pliability regime). Equation (19) is a straightforward Taylor expansion in powers of p around the point $p = 0$, where the first term is given in equation (17). In equation (19), the fact that the second term scales with τ reflects the fact that with more inertia, greater effort is required to change the emissions trajectory, in proportion to the characteristic timescale of the emitting system. The dependence on $\sqrt{p}$ shows that effort is very sensitive to p as p approaches 0. As soon as there is any transitional cost, i.e.,, for any $p > 0$, the system cannot be moved to a different starting point as in the no-pliability regime; hence, the high sensitivity to p can be explained by this qualitatively different nature of the solution. In general, the optimal initial effort increases with p, as more effort is exerted into transforming the system.[5]

4.3 Numerical illustration

Figure 2 displays the optimal abatement effort at time zero for our calibration from above, but with three values of τ, i.e., $\tau \in \{7.5, 15, 30\}$.

Optimal initial effort is increasing in p and decreasing in τ. The gains of induced innovation can easily be reaped in a flexible system with low inertia. If, however, inertial timescales put a serious brake on the optimal pace of abatement achieved, this dampens the response, and hence the benefits available, in the entire system. Policies to remove obstacles to faster transitions – many of which may be political and distributional – enhance the gains, and consequently, the justified effort.

5 Cost of delay

We end by exploring the influence of dynamic factors on the cost of delay, which we define as the sensitivity of objective function (1) to an infinitesimal delay $\mathrm{d}t$ in implementing the optimal solution given in Theorem 1. During this short period of delay, the emissions profile is assumed to equal its reference trajectory, after which the re-optimized policy is implemented. Optimization problem (1) is formally unchanged after a delay $\mathrm{d}t$ if we recognize that the initial conditions E_0 and e_0 have shifted to $E_0 + e_0\mathrm{d}t$ and $e_0 + e_1\mathrm{d}t$, respectively.[6]

First, we analytically compute (see the online Appendix B) the value of the objective function (1) under the optimal policy given in Theorem 1. We refer to this quantity as the optimal net present value (NPV) associated with problem (1). Figure 3 shows how this optimal NPV, including its three components (damages, rigid cost, and transitional cost), vary with p. The optimal NPV is decreasing in p, while climate damages make up around two thirds of the total across the range.

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[4] V is defined as follows

$$V = \left[e_1 - (e_0 - e_\star)\frac{r + z_-}{2} + (E_0 - E_\star)\frac{r\, z_-}{4}\right] \cdot \left[2(e_0 - e_\star) - z_-(E_0 - E_\star)\right]\Bigg|_{p=0,q=1},$$

$$z_\pm = r \pm \sqrt{r^2 + \frac{d}{2c}},$$

[5] A similar equation as equation (19) but for $p > p^\star$ can be obtained by plugging $p = 1$ into equation (18). As this yields no new insights, we do not display this formula. It is available upon request.

[6] For analytic tractability we choose an infinitesimal delay $\mathrm{d}t$; the results can be generalized to allow for any (non-infinitesimal) delay $\Delta t > 0$.

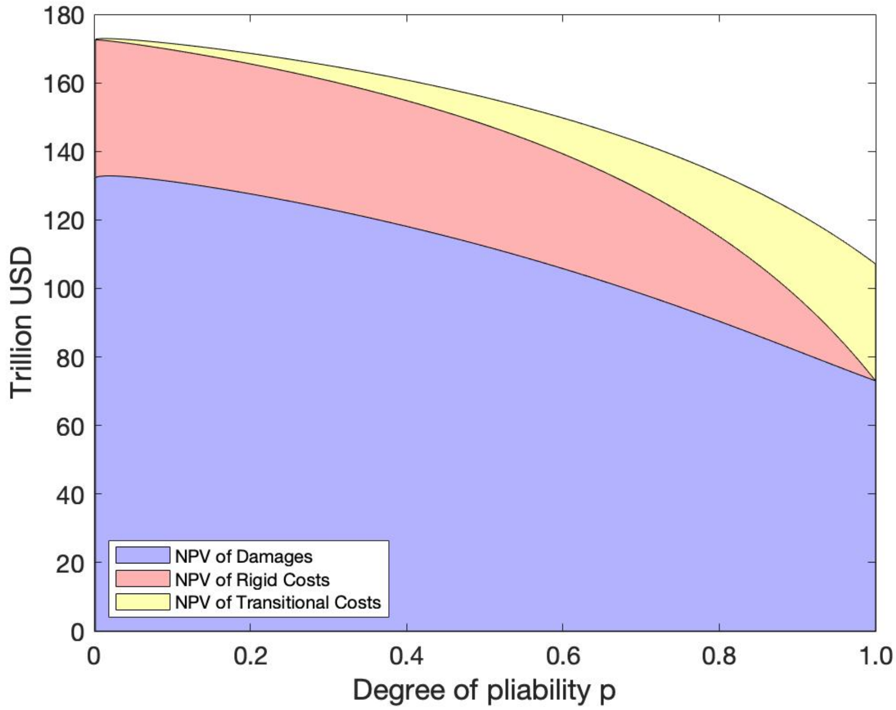

Figure 3: Decomposition of optimal value of objective function (1)

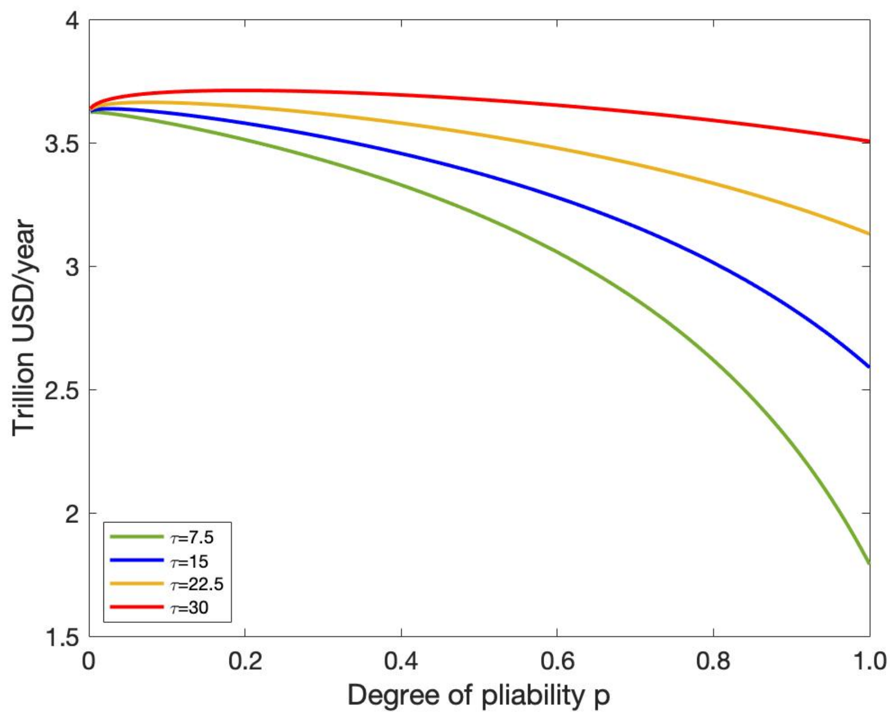

Figure 4: Cost of delay for $\tau \in \{7.5, 15, 22.5, 30\}$

Second, we analytically compute (again, see the online Appendix B) the sensitivity of the optimal NPV with respect to a short delay dt. The results shown in Figure 4 demonstrate that the cost of delay is decreasing in p from around $\sim$3.6 trillion USD per year (for $p = 0$) to around $\sim$2.6 trillion USD per year for our benchmark value $\tau = 15$. The fact that the cost of delay is a large multiple of the optimal effort at time zero suggests that the optimized objective function is highly sensitive to the initial conditions. Even as the optimal effort at time zero is relatively modest, a short delay of this optimal effort may be exceedingly costly; indeed, much more costly than the optimal effort itself.

With low pliability, climate change is more costly overall to deal with and climate damages are substantially higher. However, the system faces no inertial barrier. The ability to drop emissions immediately is valuable in terms of the large immediate marginal impact on $E(t)$, and every year that passes without such action squanders this potential, substantially increasing long-run damages. At higher pliability, abatement effort shifts towards transitional investments with enduring benefits, but the scale of (marginal) reduced climate damages is lower because the overall scale of long-run climate change is curtailed. Higher inertia, by impeding rapid response, reduces the pace at which the system can exploit lower rigid costs, but increases the marginal value of the achievable emission reductions. At a characteristic transition time of $\tau = 30$ years, these two effects roughly cancel each other out and the overall cost of delay is almost independent of the degree of system pliability.

6 Conclusion

We have constructed a stylized model that focuses on dynamic features of abatement costs, splitting the latter into rigid and transitional costs. We demonstrated that the degree to which a system depends on transitional costs – which we call pliability – has an important influence on the optimal trajectories, initial effort, and long-run economics of the system. Compared to classical formulations of abatement costs, which take into account only rigid costs but no transitional costs, systems with high pliability tend to start with optimally linear reductions in emissions, driven by higher initial effort, and result in lower long-run temperature change and damages. Independently, however, higher (lower) inertia dampens (amplifies) these benefits in our model.

Our hope is that this model will inspire further research on the dynamic features of emitting systems. Gaining a deeper understanding of different approaches to dealing with inertial timescales, induced innovation, and path dependency is crucial to help inform policymaking on one of the most important threats facing our planet.

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A Proof of Theorem 1

Euler-Lagrange equations. The optimization problem described in (1) can be solved using standard Euler-Lagrange (EL) methods. The only non-standard feature is that the control variable $E(t)$ appears alongside both of its first and second derivative in the integrand F. For this reason, the standard EL equation is adjusted to include a third term as follows

$$0 = \frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E'} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E''}. \tag{20}$$

where $F := F(E, E', E'')$ as in equation (4), where primes denote derivatives. Explicitly computing all derivatives we obtain

$$\left[-4p\tau^2 e_1 r^2 - 2q(e_0 r + e_1(rt-1)), \tfrac{d}{4c}, 2qr, 4p\tau^2 r^2 - 2q, -8rp\tau^2, 4p\tau^2\right] \begin{bmatrix} 1 \\ E(t) \\ E^{(1)}(t) \\ E^{(2)}(t) \\ E^{(3)}(t) \\ E^{(4)}(t) \end{bmatrix} = 0,$$

which we here express as an inner product involving $E(t)$ and its four derivatives. This expression makes clear that in general we are faced with an inhomogenous linear ordinary differential equation (ODE) of fourth order. As is standard, the solution can be written as the sum of two solutions: one that solves the homogenous ODE and one that solves the inhomogenous ODE.

Solution to inhomogenous ODE. The inhomogenous ODE can be solved by a linear function of time, which we write as

$$E(t) = E_\star + e_\star t, \tag{21}$$

where $E_\star$ and $e_\star$ are constants to be found. For this candidate solution $E(t)$, the second, third, and fourth derivatives are zero. Solving the resulting simplified ODE for B and b, we obtain

$$e_\star = 8\frac{c\,q\,e_1\,r}{d} \qquad\qquad E_\star = \frac{c\,q\,(e_0\,r - e_1)}{d/8} + 64\,e_1\,r^2\left(\frac{c\,p\,\tau^2}{4\,d} - \left(\frac{c\,q}{d}\right)^2\right). \tag{22}$$

This simple solution already yields one important insight into the long-term behavior of our solution: in the long run, optimal cumulative emissions are linear in time, such that annual emissions are optimally constant. Specifically, the optimal long-run constant level of emissions is given by the parameter $e_\star$ above. As can be seen, it decreases with the damage parameter d, but increases with the rigid-cost component q, the discount rate r, and the increase of marginal emissions in the reference scenario, given by e_1.

While the inhomogenous ODE determines the optimal long-term emissions path, the particular solutions to the homogeneous ODE determine the optimal course of action in the short-term. We discuss these next.

Solution to homogenous ODE. To solve the homogenous ODE, we look for solutions that are exponential in time. Indeed, the homogenous ODE of fourth order allows for four independent solutions taking the form

$$E(t) = \sum_{j=1}^{4} Z_j \exp\left(\frac{z_j\, t}{2}\right), \tag{23}$$

where the parameters Z_j and z_j remain to be determined for $j = 1, 2, 3, 4$. Substituting this candidate solution into the ODE and simplifying, we find that the constants z_j for each $j = 1, 2, 3, 4$ must solve the following fourth-order polynomial equation

$$\left[d/c, 4\,q\,r, 4\,r^2\,p\,\tau^2 - 2\,q, -4\,r\,p, p\,\tau^2\right] \begin{bmatrix} 1 \\ z_j \\ z_j^2 \\ z_j^3 \\ z_j^4 \end{bmatrix} = 0, \qquad j = 1, 2, 3, 4. \tag{24}$$

Generally, this equation is of fourth order, unless $p = 0$, in which case it is only of second order (note the last two entries of the row vector).

Full solution. The full solution is obtained by summing the solutions to the homogeneous and inhomogeneous ODEs, i.e.,

$$E(t) = E_\star + e_\star\, t + \sum_{j=1}^{4} Z_j \exp\left(\frac{z_j t}{2}\right), \tag{25}$$

where the parameters $E_\star$ and $e_\star$ are given by (22), the constants z_j for $j = 1, 2, 3, 4$ are the roots of the fourth-order polynomial equation given in (24), and the four constants Z_j for $j = 1, 2, 3, 4$ remain to be determined by four boundary conditions, as discussed below. These boundary conditions will need to ensure that $E(0) = E_\star + \sum_{j=1}^{4} Z_j = E_0$, thereby putting a constraint on the Z_j's.

Boundary conditions. In general, the four constants Z_j are determined by a total of four boundary conditions to be specified at either $t = 0$ or $t = T$. At $t = 0$, we impose $E(0) = E_0$, reflecting the fact that cumulative emissions (relative to pre-industrial times) at time zero are fixed. For systems with any positive transitional cost ($p > 0$), we also impose $E'(0) = E'_{\text{ref}}(0) = e_{\text{ref}}(0) = e_0$, because sudden jumps in marginal emissions would incur an infinite cost. By imposing both boundary conditions, we ensure that the path of cumulative emissions $E(t)$ smoothly matches that of the reference trajectory of cumulative emissions $E_{\text{ref}}(t)$.

At $t = T$, we are faced with two free boundary conditions, as endpoint $E(T)$ and its derivative $E'(T)$ are left to be determined by the optimizer. However, in the limit as $T \to \infty$, which we consider below, two of the four homogenous solutions can be discarded (set to zero), as they blow up exponentially, thereby causing infinite damages. As such, only two constants Z_j, for $j = 1, 2$ remain, which can be determined by the two boundary conditions at $t = 0$.

If $p = 0$, the ODE and polynomial equation are of second order. In this case, only a single boundary condition at $t = 0$ is required, which we take to be $E(0) = E_0$. In this case, a jump in marginal (but not cumulative) emissions at time zero is permitted.

Solutions under three regimes. The optimal solution behaves differently, qualitatively, depending on the numerical values of the parameters. Specifically, three regimes can be identified. We present the solution in each of three mutually exclusive and collectively exhaustive regimes:

1. No pliability: $p = 0$,
2. Medium pliability: $0 < p \leq p_\star$, which implies $c\,q^2 \geq p\,\tau^2\,d$,
3. High pliability: $p > p_\star$, which implies $c\,q^2 < p\,\tau^2\,d$.

The critical boundary between the medium- and high-pliability regimes is denoted $p^\star$ and is determined by setting p equal to $p^\star$, q equal to $1-p^\star$ and solving for $p^\star$ the equality $c\,q^2 = p\,\tau^2\,d$, i.e., we must solve

$$c\,(1-p^\star)^2 \;=\; p^\star \tau^2 d.$$

This is a quadratic equation in $p^\star$ with two potential solutions. Only one of these potential solutions falls in the range $(0,1)$, which reads

$$p^\star := 1 - \frac{\sqrt{1+4x}-1}{2x} \in (0,1),$$

where $x := c/(d\tau^2) \in (0,\infty)$ is a dimensionless characteristic of the system. For $0 < p \leq p^\star$, it can be verified that $c\,q^2 \geq p\,\tau^2\,d$, such that we are in the medium-pliability regime. For $p > p^\star$, we can be verified that $c\,q^2 < p\,\tau^2\,d$, such that we are in the high-pliability regime.

In each case, an analytic solution is possible, which can be found by (i) solving the (in general) fourth-order polynomial equation, (ii) discarding two of the four solutions to the homogeneous ODE that correspond to the explosive solutions, and (iii) imposing the relevant boundary condition(s) at $t = 0$. We here only report the analytic solution in the case where $T = \infty$, which is economically the most relevant, and for which the solution takes the simplest possible form.

Zero pliability: If $p = 0$, such that the system contains no pliability, the fourth-order ODE simplifies to a second-order ODE. The corresponding second-order polynomial equation allows for two unique roots, one positive and one negative. The positive root can be discarded as it corresponds to an explosive solution, such that we can set $Z_2 = Z_3 = Z_4 = 0$, leaving only Z_1 to be determined. The negative root is given by

$$z_1 = r - \sqrt{r^2 + \frac{d}{2\,c\,q}}. \tag{26}$$

Note that $z_1 < 0$; the other root contains a plus instead of a minus in front of the square root and is economically irrelevant. This confirms the first part of equation (12) in Theorem 1. Imposing the boundary conditions $E(0) = E_0$, the constant Z_1 can be determined as

$$Z_1 = E_0 - E_\star, \tag{27}$$

where the value of $E_\star$ is given by (22) when p is set to zero. This confirms the second part of equation (12) in Theorem 1. For the zero pliability regime, we do not impose $E'(0) = e_0$ such that the optimal level of today's emissions, $E'(0)$, will generally differ from the reference level, e_0. For pliable systems in the two regimes below, a jump in marginal emissions is impossible.

Medium pliability: If $p \neq 0$, $c\,q^2 \geq p\,\tau^2\,d$, such that pliability is non-zero but small in relative terms (i.e., $0 < p \leq p^\star$), the fourth-order polynomial allows for four distinct roots. Two roots are positive and can be discarded from economic arguments, i.e., we set $Z_3 = Z_4 = 0$. The two remaining (negative) roots are given by

$$z_1 = r - \sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\left(\frac{q}{p\,\tau^2}\right)^2 - \frac{d}{c\,p}}}, \tag{28}$$

$$z_2 = r - \sqrt{r^2 + \frac{q}{p\,\tau^2} - \sqrt{\left(\frac{q}{p\,\tau^2}\right)^2 - \frac{d}{c\,p\,\tau^2}}}, \tag{29}$$

where each displayed square root is a real number because $cq^2 \geq p\,\tau^2\,d$ by assumption in the current regime, which implies $(q/p\tau^2)^2 \geq d/(cp)$. These equations confirm equations (13) in Theorem 1.

Imposing the boundary conditions $E(0) = E_0$ and $E'(0) = e_0$, we find the two constants Z_1 and Z_2 as follows

$$Z_1 = \frac{2(e_0 - e_\star) + z_2(E_\star - E_0)}{z_1 - z_2} \qquad\qquad Z_2 = \frac{2(e_0 - e_\star) + z_1(E_\star - E_0)}{z_2 - e_1}. \tag{30}$$

These equations confirm equations (14) in Theorem 1.

High pliability: If $p \neq 0$, $c\,q^2 < p\,\tau^2\,d$, such that transitional costs are large in relative terms (i.e., $p > p^\star$), the fourth-order polynomial equation allows for four distinct, complex-valued, roots. To avoid the emissions path exploding as $t \to \infty$, we pick the two roots with negative real parts. Hence, we may set $Z_3 = Z_4 = 0$. The two negative roots z_1 and z_2 differ by only a single sign, such that we can denote them by $z_1 = z_+$ and $z_2 = z_-$, where $z_\pm$ is defined as

$$\begin{aligned} z_\pm \equiv r - \frac{1}{\sqrt{2}}\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}} \\ \pm \frac{\mathrm{i}}{\sqrt{2}} \frac{\sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2}}{\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}}}, \end{aligned} \tag{31}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, and every displayed square root is a real (positive) number, because $c\,q^2 < p\tau^2 d$ in the current regime. It is clear that both $z_\pm$ have negative real parts as desired. This confirms equation (15) in Theorem 1.

Imposing the boundary conditions $E(0) = E_0$ and $E'(0) = e_0$, we find that the constants Z_1 and Z_2 are identical in form to those in the medium-pliability regime, namely

$$Z_1 = \frac{2(e_0 - e_\star) + z_2(E_\star - E_0)}{z_1 - z_2} \qquad\qquad Z_2 = \frac{2(e_0 - e_\star) + z_1(E_\star - E_0)}{z_2 - z_1}. \tag{32}$$

However, the numerical values of these constants differ from those in the medium pliability regime, because the two roots z_1 and z_2, which appear in the numerator and denominator, are now complex values. Hence, Z_1 and Z_2 are also complex valued. Naturally, the cumulative emissions path $E(t)$ for all time t remains real valued. After some tedious but straightforward trigonometric algebra, the optimal cumulative emissions trajectory $E(t)$ can be rewritten in trigonometric terms as

$$E(t) = E_\star + e_\star t + \exp\left(\frac{\widehat{z}t}{2}\right)\left[\frac{2(e_0 - e_\star) + \widehat{z}(E_\star - E_0)}{\widetilde{z}} \sin\left(\frac{\widetilde{z}t}{2}\right) + (E_0 - E_\star)\cos\left(\frac{\widetilde{z}t}{2}\right)\right],$$

where $e_\star$ and $E_\star$ are as in (22), while $\widehat{z}$ and $\widetilde{z}$ are real numbers coming from the real and imaginary parts of z_1 above. Explicitly, we have

$$\widehat{z} = r - \frac{1}{\sqrt{2}}\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}} \tag{33}$$

and

$$\widetilde{z} = \frac{1}{\sqrt{2}} \frac{\sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2}}{\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}}}. \tag{34}$$

The intuition for the high pliability regime is that, when "steering" is expensive, it might be beneficial to "oversteer" before correcting (steering back) later, which explains the appearance of trigonometric functions in the solution: emissions oscillate towards the long-term optimum. For a fully pliable system in which case $q = 0$, it is optimal to decarbonize the economy completely at some finite time, and even go into negative marginal emissions (capturing carbon dioxide from the atmosphere), also at some finite time, while oscillating (with exponentially decreasing amplitudes) towards a fully decarbonized limit.

In all three regimes, the optimal marginal emissions path $E'(t)$ is implied by the optimal cumulative emissions path $E(t)$ via a straightforward differentiation with respect to time. Further, in all cases an analytic solution remains possible even for a finite optimization horizon T, but the resulting expressions are more involved, because it no longer holds that two out of four roots from the fourth-order polynomial can be discarded (all four roots are relevant in this case). The resulting expressions are available from the authors upon request.

B Analytic solution for NPV and cost of delay

Assume $0 < p \leq p^\star$, such that the medium-pliability regime applies; below we extend the results to all $p \in [0, 1]$. Assume $T = \infty$, i.e., an infinite time horizon. Assume the optimal cumulative emissions path $E(t)$ given in equation (10) in Theorem 1. Then the net present value (NPV) of damages can be computed analytically as

$$\int_0^\infty \exp(-r\,t)\,\frac{d}{8}E(t)^2\,\mathrm{d}t = \tag{35}$$
$$\frac{d}{8}\left[\frac{2e_\star^2 + 2e_\star E_\star r + E_\star^2 r^2}{r^3} + \frac{4Z_1Z_2}{2r - z_1 - z_2} + \sum_{i=1}^{2}\left\{\frac{Z_i^2}{r - z_i} + \frac{8Z_i(e_\star + E_\star r)}{(2r - z_i)^2} - \frac{4z_iZ_iE_\star}{(2r - z_i)^2}\right\}\right].$$

Second, the NPV of the rigid-cost component can be computed in closed form as

$$\int_0^\infty \exp(-r\,t)\,c\,q\,[e_{\rm ref}(t) - e(t)]^2\,{\rm d}t = \tag{36}$$

$$\frac{c\,q}{4}\left[\frac{8e_1^2}{r^3} + \frac{4(e_\star - e_0)^2}{r} + 8e_1\frac{e_0 - e_\star}{r^2} + \frac{4z_1z_2Z_1Z_2}{2r - z_1 - z_2}\right.$$
$$\left. + \sum_{i=1}^{2}\left\{8(e_\star - e_0)\frac{z_iZ_i}{2r - z_i} + \frac{z_i^2Z_i^2}{r - z_i} - 8e_1\frac{2z_iZ_i}{(2r - z_i)^2}\right\}\right].$$

Third, the NPV of the transitional-cost component reads

$$\int_0^\infty \exp(-r\,t)\,2\,c\,p\,\tau^2\,[e'_{\rm ref}(t) - e'(t)]^2\,{\rm d}t = \tag{37}$$
$$\frac{cp\tau^2}{8}\left[\frac{16e_1^2}{r} + \frac{z_1^4Z_1^2}{r - z_1} + \frac{z_2^4Z_2^2}{r - z_2} + \frac{4z_1^2z_2^2Z_1Z_2}{2r - z_1 - z_2} - 16e_1\left(\frac{z_1^2\,Z_1}{2r - z_1} + \frac{z_2^2\,Z_2}{2r - z_2}\right)\right].$$

In equations (35), (36) and (37), the quantities $e_\star$, $E_\star$ are given in equation (11), while z_i for $i = 1, 2$ are given in equation (13), and Z_i for $i = 1, 2$ are given in equation (14).

By adding the right-hand side (RHS) of equations (35), (36) and (37), we obtain the optimal NPV of the entire minimisation problem (1), i.e.,

$$\text{NPV} = \text{RHS of equations (35), (36) and (37).} \tag{38}$$

This optimal NPV remains valid in the limit where p approaches zero, such that the NPV in the no-pliability regime can be obtained as a special case. Moreover, all expressions technically remain valid in the high-pliability regime; while some quantities turn complex, the imaginary parts cancel out and the result is a real-valued number that equals the desired NPV in the high-pliability regime.

The cost of delay discussed in the main text is obtained by comparing the NPV as computed above with the NPV evaluated a small time ${\rm d}t$ later, assuming no action is taken in the meanwhile. Our solution remains valid after some delay if we recognise that the initial conditions have shifted. In particular, cumulative emissions have increased from E_0 to $E(0 + {\rm d}t) = E_0 + e_0{\rm d}t$, while annual emissions have increased from e_0 to $e(0 + {\rm d}t) = e_0 + e_1{\rm d}t$. Hence, with obvious notation,

$$\text{cost of delay} \;=\; \frac{{\rm d}\,\text{NPV}}{{\rm d}\,E_0}\,e_0 \;+\; \frac{{\rm d}\,\text{NPV}}{{\rm d}\,e_0}\,e_1, \tag{39}$$

where the NPV is given in equation (38). The cost of delay is measured in units of currency per units of time. The required derivatives can be computed in closed form by using equations (35), (36) and (37), which depend explicitly on E_0 and e_0. Moreover, the chain rule must be employed to account for the implicit dependence of $E_\star$, Z_1 and Z_2 on the initial conditions E_0 and e_0; the resulting (lengthy) expression for equation (39) is available from the authors on request.